where the plus sign in the subscripts applies for $j = 1$ and the minus sign for $j = 2$;

3) the same boundary conditions as the functions $u_j(x, t)$;

4) the same (but homogeneous) linear matching conditions as the functions $u_j(x, t)$ at the sites of the concentrated masses, rigid or elasticomassive supports with linear characteristics, etc.

It is easy to show that relations (2.12) are equations in variations for nonlinear matching conditions (2.2) for $x = l_r$. Unlike the remaining boundary conditions for the perturbations ξ_j, (2.12) contains terms with π / ω-periodic coefficients. If the modulation of these coefficients is not large, then characteristic exponents (1.15) can be determined by the method of Sect. 1 with allowance for the appended Note.

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PROBLEMS OF OPTIMIZATION WITH CONSTRAINTS IMPOSED ON THE PHASE COORDINATES

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We consider the problems of optimization of control processes with first and higher order constraints imposed on the phase coordinates [1–3]. We establish conditions which make easier the determination of the point at which the phase trajectory leaves the boundary of the region of admissible variation of coordinates.

1. Statement of the problem. The problem studied in [2, 4] was the following. Out of the continuous functions $x_s(t)$, $(s = 1,\ldots, n)$ possessing piece-wise continuous derivatives $x_s^{\cdot}(t)$ and out of the piece-wise continuous controls $u_k(t)$, $(k = 1,\ldots, m)$ satisfying the differential equations

$$g_s = x_s^{\cdot} - f_s(x, u, t) = 0 \qquad (s = 1,\ldots n) \tag{1.1}$$

on the interval $[t_0, T]$, the relations

$$\psi_k = \psi_k(x, u, t) = 0 \quad (k = 1,\ldots, r < m) \tag{1.2}$$

the inequality

$$\vartheta(x) \leqslant 0 \tag{1.3}$$

at the ends of the segment $[t_0, T]$ and the conditions

$$\varphi_l = \varphi_l[x(t_0), t_0, x(T), T] = 0 \ (l = 1,\ldots, p \leqslant 2n + 1) \tag{1.4}$$

to find those, which minimize the functional

$$I = g[x(t_0), t_0, x(T), T] + \int_{t_0}^{T} f_0(x, u, t)\, dt \tag{1.5}$$

Here x and u denote the respective sets of phase coordinates $x_1,\ldots x_n$ and controls $u_1,\ldots u_m$.

In such problems the optimal trajectory may include segments belonging to the boundary of the region defined by the inequality (1.3). In the following, we shall concentrate our attention on such segments.

If a segment of the trajectory lying on the interval $[t_1, t_2]$ belongs to the boundary of the region (1.3), we can describe this fact in two different ways. Firstly, we can assume that $\vartheta = 0$ for $t \in [t_1, t_2]$, stressing the fact that when $t = t_1$, $\vartheta = 0$. Secondly, we can say that for $t = t_1$, $\vartheta = 0$ and the following equation

$$\vartheta_1 = \frac{d\vartheta}{dt} = \sum_{s=1}^{n} \frac{\partial \vartheta}{\partial x_s} f_s = 0 \qquad (1.6)$$

is satisfied for $t \in [t_1, t_2]$. The latter may take place only when ϑ_1 depends on the control parameters explicitly, so that $\vartheta_1 = \vartheta_1(x, u, t)$. The constraints possessing this property are usually called the first order coordinate constraints [3].

If the function

$$\vartheta_1, \vartheta_2 = \frac{d\vartheta_1}{dt} = \frac{\partial \vartheta_1}{\partial t} + \sum_{s=1}^{n} \frac{\partial \vartheta_1}{\partial x_s} f_s, \ldots, \vartheta_{q-1} = \frac{d\vartheta_{q-2}}{dt} = \frac{\partial \vartheta_{q-2}}{\partial t} + \sum_{s=1}^{n} \frac{\partial \vartheta_{q-2}}{\partial x_s} f_s$$

does not depend explicitly on the controls and the function

$$\vartheta_q = \frac{\partial \vartheta_{q-1}}{\partial t} + \sum_{s=1}^{n} \frac{\partial \vartheta_{q-1}}{\partial x_s} f_s$$

does, the constraints are called the qth order coordinate constraints [3].

2. First order constraints. If the first method is used to describe the boundary of the region (1.3) and we assume that $\vartheta = 0$ when $t \in [t_1, t_2]$ and $t = t_1$, the functional I introduced in [2] has the form

$$I = \varphi + \int_{t_0}^{T} \Big[\sum_{s=1}^{n} \lambda_s^{(1)} x_s^{\cdot} - H^{(1)} - \alpha_0 (\vartheta + u_{m+1}^2)\Big] dt + \nu_0 \vartheta [x(t_1)] \qquad (2.1)$$

$$H^{(1)} = -f_0 + \sum_{s=1}^{n} \lambda_s^{(1)} f_s + \sum_{k=1}^{r} \mu_k^{(1)} \psi_k, \qquad \varphi = g + \sum_{l=1}^{p} \rho_l \varphi_l \qquad (2.2)$$

An additional control u_{m+1} is introduced here together with the auxiliary relation $\vartheta + u_{m+1}^2 = 0$, the latter allowing for the boundedness of the region of admissible variation of the phase coordinates.

The second method of describing the boundary leads to the following functional:

$$I = \varphi + \int_{t_0}^{T} \Big[\sum_{s=1}^{n} \lambda_s^{(2)} x_s^{\cdot} - H^{(2)} - \alpha_1 \vartheta_1\Big] dt + \nu_1 \vartheta [x(t_1)] \qquad (2.3)$$

The function $H^{(2)}$ appearing in it can be obtained from the first relation of (2.2) by making the relevant changes in the superscript.

Constructing the necessary conditions of stationarity and performing the routine variational operations, we obtain the following equations for the first case:

$$\lambda_s^{\cdot(1)} = \frac{\partial f_0}{\partial x_s} - \sum_{i=1}^{n} \lambda_i^{(1)} \frac{\partial f_i}{\partial x_s} - \sum_{i=1}^{r} \mu_i^{(1)} \frac{\partial \psi_i}{\partial x_s} - \alpha_0 \frac{\partial \vartheta}{\partial x_s} \quad (s = 1, \ldots, n) \qquad (2.4)$$

$$-\frac{\partial f_0}{\partial u_k} + \sum_{s=1}^{n} \lambda_s^{(1)} \frac{\partial f_s}{\partial u_k} + \sum_{i=1}^{r} \mu_i^{(1)} \frac{\partial \psi_i}{\partial u_k} = 0 \quad (k = 1, \ldots, m)$$

and

$$\lambda_s^{\cdot(2)} = \frac{\partial f_0}{\partial x_s} - \sum_{i=1}^{n} \lambda_i^{(2)} \frac{\partial f_i}{\partial x_s} - \sum_{i=1}^{r} \mu_i^{(2)} \frac{\partial \psi_i}{\partial x_s} - \alpha_1 \frac{\partial \vartheta_1}{\partial x_s} \quad (s = 1, \ldots, n) \qquad (2.5)$$

$$-\frac{\partial f_0}{\partial u_k}+\sum_{s=1}^{n}\left(\lambda_s^{(2)}+\alpha_1\frac{\partial\vartheta}{\partial x_s}\right)\frac{\partial f_s}{\partial u_k}+\sum_{i=1}^{r}\mu_i^{(2)}\frac{\partial\psi_i}{\partial u_k}=0\quad(k=1,\ldots,m)\qquad\text{(cont.)}$$

for the second case. In the latter we use the relation

$$\frac{\partial\vartheta_1}{\partial u_k}=\sum_{s=1}^{n}\frac{\partial\vartheta}{\partial x_s}\frac{\partial f_s}{\partial u_k}\qquad(k=1,\ldots,m)\tag{2.6}$$

and write only the part of the expanded stationarity condition, which shall be used subsequently.

The solution of the optimization problem is independent of the method used to describe the boundary segments of the integral curves. Functions x_s and u_k satisfying (2.4) and (2.5) should not differ from each other. To fulfil this condition, it is necessary that

$$\lambda_s^{(1)}=\lambda_s^{(2)}+\alpha_1\frac{\partial\vartheta}{\partial x_s}\quad(s=1,\ldots,n),\qquad\mu_k^{(1)}=\mu_k^{(2)}\quad(k=1,\ldots,r)$$

$$\alpha_0=-\frac{d\alpha_1}{dt}\tag{2.7}$$

hold. The last of these equations is particularly important. We note at this stage that analysis of the conjugation conditions for the points $t=t_1$ and $t=t_2$ which are given by

$$\lambda_s^{(1)}(t_1-0)=\lambda_s^{(1)}(t_1+0)-\nu_0(\partial\vartheta/\partial x_s)_{t_1}$$

$$\lambda_s^{(2)}(t_1-0)=\lambda_s^{(2)}(t_1+0)-\nu_1(\partial\vartheta/\partial x_s)_{t_1}$$

$$\lambda_s^{(1)}(t_2-0)=\lambda_s^{(1)}(t_2+0),\qquad\lambda_s^{(1)}(t_1-0)=\lambda_s^{(2)}(t_1-0)$$

$$\lambda_s^{(2)}(t_2-0)=\lambda_s^{(2)}(t_2+0),\qquad\lambda_s^{(1)}(t_2+0)=\lambda_s^{(2)}(t_2+0)$$

leads to the relations

$$\alpha_1(t_2)=0,\qquad\alpha_1(t_1)=\nu_0-\nu_1\tag{2.8}$$

Let us now consider the Clebsch inequality. It can easily be shown that, when the first method of describing the boundary segments is used, the inequality assumes the form

$$\sum_{i=1}^{m}\sum_{k=1}^{m}\frac{\partial^2H^{(1)}}{\partial u_i\partial u_k}\delta u_i\delta u_k+2\alpha_0(\delta u_{m+1})^2\leqslant 0$$

On setting $\delta u_i=0$ $(i=1,\ldots,m)$ and $\delta u_{m+1}\neq 0$, we obtain $2\alpha_0(\delta u_{m+1})^2\leqslant 0$, which in turn yields

$$\alpha_0\leqslant 0\tag{2.9}$$

Using the last relation of (2.7), we obtain by (2.9)

$$d\alpha_1/dt\geqslant 0\tag{2.10}$$

which, together with (2.8), may be found useful in determining the instant $t=t_2$ at which the optimal trajectory leaves the boundary of the region (1.3).

3. Constraints of qth order. When the qth order constraints appear in the problem, the boundary segments of the optimal trajectory can be described in $q+1$ equivalent ways. Indeed, the condition that j equations $\vartheta_i=0$ $(i=0,1,\ldots,j\leqslant q-1)$ $(\vartheta_0=\vartheta)$ hold for $t=t_1$ and only relations $\vartheta_{j+1}=0$ when $t\in[t_1,t_2]$, is sufficient for the representative point to lie on the interval $[t_1,t_2]$ of the boundary of the region (1.3).

In each of these methods the functional I has the form

$$I_i = \varphi + \int_{t_0}^{T} \Big[\sum_{s=1}^{n} \lambda_s^{(i)} x_s^{\cdot} - H^{(i)} - \alpha_i \vartheta_i\Big] dt + \sum_{k=}^{i-1} \nu_k \vartheta_k [x(t_1), t_1] \qquad (i = 1, \ldots, q) \tag{3.1}$$

which reduces to (2.1) for $i = 0$. Formulas defining $H^{(i)}$ are obtained from the first relation of (2.2) by the appropriate change of the superscripts. Expanding the necessary condition of stationarity, we find the following equations:

$$\lambda_s^{(i)\cdot} = \frac{\partial f_0}{\partial x_s} - \sum_{k=1}^{n} \lambda_k^{(i)} \frac{\partial f_k}{\partial x_s} - \sum_{k=1}^{r} \mu_k^{(i)} \frac{\partial \psi_k}{\partial x_s} - \alpha_i \frac{\partial \vartheta_i}{\partial x_s} \qquad (s = 1, \ldots, n)$$

$$-\frac{\partial f_0}{\partial u_k} + \sum_{s=1}^{n} \Big(\lambda_s^{(i)} + \alpha_i \frac{\partial \vartheta_{i-1}}{\partial x_s}\Big) \frac{\partial f_s}{\partial u_k} + \sum_{s=1}^{r} \mu_s^{(i)} \frac{\partial \psi_s}{\partial u_k} = 0 \qquad (k = 1, \ldots, m) \tag{3.2}$$

Equations corresponding to the value $i = 0$ have the form of (2.4).

When analyzing Eqs. (3.2), we should recall that the definition of the qth order constraints implies that

$$\frac{\partial \vartheta_i}{\partial u_k} = 0 \quad (i = 0, \ldots, q-1), \qquad \frac{\partial \vartheta_q}{\partial u_k} = \sum_{s=1}^{n} \frac{\partial \vartheta_{q-1}}{\partial x_s} \frac{\partial f_s}{\partial u_k}$$

hold. Consequently, repeating the operations described in Sect. 2 we shall arrive at the following relations:

$$\lambda_s^{(i)} = \lambda_s^{(i+1)} + \alpha_{i+1} \frac{\partial \vartheta_i}{\partial x_s} \quad (s = 1, \ldots, n) \qquad \mu_k^{(i)} = \mu_k^{(i+1)} \quad (k = 1, \ldots, r)$$

$$\alpha_i = -\frac{d\alpha_{i+1}}{dt} \qquad (i = 0, \ldots, q-1) \tag{3.3}$$

When the latter hold, all solutions of the optimal problem, using any of the methods of describing the boundary segments given above, will coincide. When $t = t_2$, the equation

$$\alpha_i (t_2) = 0 \quad (i = 1, \ldots, q) \tag{3.4}$$

should hold. From (3.3) we have

$$\alpha_i = (-1)^{q-i} \frac{d^{q-i} \alpha_q}{dt^{q-i}}$$

Using again the Clebsch inequality and conditions (1.3), we obtain the following set of inequalities:

$$(-1)^i \frac{d^i \alpha_q}{dt^i} \leqslant 0 \quad (i = 0, \ldots, q) \tag{3.5}$$

The latter should be used for investigating the boundary segments with the qth order constraints.

E x a m p l e. Let us consider the optimization problem for

$$x_1^{\cdot} = x_2 u_1, \qquad x_2^{\cdot} = {}^1\!/_2\, u_2 \tag{4.1}$$

and the relation

$$\psi (u_1, u_2) = u_1^2 + u_2^2 - 1 = 0 \tag{4.2}$$

We require that the inequality

$$\vartheta (x_1, x_2) = x_2^2 - x_1 \operatorname{tg} \theta - \frac{2\alpha}{\pi} \Big[1 - \Big(\frac{\pi}{2} - \theta\Big) \operatorname{tg} \theta\Big] \leqslant 0 \tag{4.3}$$

and conditions

$$x_1 (0) = x_2 (0) = 0 \tag{4.4}$$

hold.

We seek u_1 and u_2 which minimize the time T in which the system reaches the line

$x_1 = 1$. The brachistochrone problem could be stated in this form under the condition that the point cannot descend below a certain straight line.

In the inequality (4.3), α and θ have constant values falling within the limits $0 \leqslant \alpha \leqslant 1$ and $0 \leqslant \theta \leqslant {}^1/_2\pi$.

By constructing the functional I corresponding to the second method of describing the boundary and by making use of the equation and relationships pertinent to this variational problem, we arrive at the following solution. On the interval $0 \leqslant t \leqslant t_1$ corresponding to the interior points of the region (1.3) we have

$$\lambda_1 = C_1, \qquad x_1 = {}^1/_2\, C_1^{-2}\,(C_1 t - \sin C_1 t) \quad u_1 = \sin {}^1/_2\, C_1 t; \quad \mu_1 = -1$$

$$\lambda_2 = 2\cos {}^1/_2\, C_1 t, \qquad x_2 = C_1^{-1} \sin {}^1/_2\, C_1 t, \quad u_2 = \cos {}^1/_2\, C_1 t \tag{4.5}$$

On the boundary interval $t_1 \leqslant t \leqslant t_2$ the solution has the form

$$\lambda_1 = C_2, \qquad x_1 = \frac{1}{4}\,\frac{k}{1+k^2}\,t^2 + A_1 (1+k^2)^{-1/2}\, t + A_2, \qquad u_1 = (1+k^2)^{-1/2}$$

$$\lambda_2 = -C_2 (1+k^2)^{-1/2}\, t + A_3, \quad x_2 = {}^1/_2\, k\,(1+k^2)^{-1/2}\, t + A_1, \quad u_2 = k\,(1+k^2)^{-1/2} \tag{4.6}$$

$$\mu_1 = -{}^1/_2, \qquad \nu_1 = \frac{C_1 - C_2}{k}, \qquad \alpha_1 = [k - {}^1/_2\,\lambda_2 (1+k^2)^{1/2}]\, x_2^{-1} (1+k^2)^{-1/2}$$

Finally, on the interval $t_2 \leqslant t \leqslant T$ on which the representative point is again interior to (1.3), the solution is

$$\lambda_1 = C_2, \quad x_1 = {}^1/_2\, C_2^{-2}\,[C_2 t - \sin C_2 (T - t)] + C_3, \quad u_1 = \cos {}^1/_2\, C_2 (T - t)$$

$$\mu_1 = -1 \tag{4.7}$$

$$\lambda_2 = 2 \sin {}^1/_2\, C_2 (T - t), \qquad x_2 = C_2^{-1} \cos {}^1/_2\, C_2 (T - t), \qquad u_2 = \sin {}^1/_2\, C_2 (T - t)$$

where

$$t_1 = 2\,(\alpha h)^{1/2}\,({}^1/_2\,\pi - \operatorname{arctg} k)\,(1 - {}^1/_2\, k\,(\pi - 2 \operatorname{arctg} k)]^{-1/2}$$

$$t_2 = \frac{2}{k}\left[\left(\frac{1 + \alpha h/k}{k^{-1} + \operatorname{arctg} k}\right)^{1/2} - \left(\frac{\alpha h}{k}\,(k^{-1} - {}^1/_2\,\pi + \operatorname{arctg} k)\right)^{1/2}\right]$$

$$T = 2\,\{[(1 + \alpha h/k)\,(k^{-1} + \operatorname{arctg} k)]^{1/2} - [(\alpha h/k)\,(k^{-1} - {}^1/_2\,\pi + \operatorname{arctg} k)]^{1/2}\}$$

$$C_1 = \left[\frac{1 - {}^1/_2\, k\,(\pi - 2 \operatorname{arctg} k}{\alpha h}\right]^{1/2}, \qquad C_2 = \left(\frac{k^{-1} + \operatorname{arctg} k}{1 + \alpha h/k}\right)^{1/2}$$

$$C_3 = \left[(\alpha h/k)\,\frac{1 + \alpha h/k}{k^{-1} + \operatorname{arctg} k}\,(k^{-1} - {}^1/_2\,\pi + \operatorname{arctg} k)\right]^{1/2} - \alpha h/k$$

$$A_1 = \left\{\frac{\alpha h\,[1 - {}^1/_2\, k\,(\pi - 2 \operatorname{arctg} k)]}{1 + k^2}\right\}^{1/2}, \qquad A_2 = -\alpha h\,\frac{{}^1/_2\,\pi - \operatorname{arctg} k + k}{1 + k^2}$$

$$A_3 = \frac{2}{k}\left\{(1 + k^2)^{1/2} - \left[\frac{\alpha h\,(k^{-1} + \operatorname{arctg} k)\,[1 - {}^1/_2\, k (\pi - 2 \operatorname{arctg} k)]}{(1 + \alpha h/k)\,(1 + k^2)}\right]^{1/2}\right\}$$

$$k = tg\,\theta, \qquad h = 2\pi^{-1}\,[1 - ({}^1/_2\,\pi - \theta)\; tg\,\theta] \tag{4.8}$$

Inserting λ_2 and x_2 into the last relation of (4.6) and differentiating with respect to time we easily find, that

$$d\alpha_1/dt > 0 \qquad \text{for } t \in [t_1, t_2], \qquad \alpha_1 (t_2) = 0 \tag{4.9}$$

which coincide with those obtained in Sect. 2.

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SOME PROPERTIES OF PLANE AND THREE-DIMENSIONAL BODIES

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We determine the relationships between the structural elements of geometric bodies represented as plane or three-dimensional systems in the form of hinged-rod or rigid hingeless lattices by considering the intersections and gaps of the latter in a plane field. Three-dimensional bodies can be investigated by projecting them on a plane. The projections considered in the present paper exclude complete coincidence of individual elements. One way of establishing the relationships between the elements of plane and three-dimensional bodies is by mathematical induction.

1. Initial assumptions. We begin by defining some terms:

Rods are straight or curvilinear bodies one of whose dimensions is large compared to the other. These bodies possess three degrees of freedom in a plane and five degrees of freedom in space.

Disks are bodies or geometrically nonvarying links with three degrees of freedom in a plane and six degrees of freedom in space.

Hingeless (free) intersections are domains or points of contact between elements (disks, rods, or both).

Hinged intersections are domains or points of contact between elements into which hinges have been introduced.

Gaps are individual closed domains within the outer contour whose dimensions can be determined by direct calculation of the ares of clearance.

We assume that the gaps can be of any shape, e. g. a uniangle (a domain bounded by a closed curve with a single acute or obtuse corner), a biangle (a domain bounded by a closed curve. with two acute or obtuse corners or one of each), a triangle, a polygon, or a nonangle (a domain bounded by a closed curve with smooth transitions from one curve to another).